Invertible Quadratic Non-Linear Layers for MPC-/FHE-/ZK-Friendly Schemes over $\mathbb{F}_p^n$

Application to Poseidon

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Motivation

Symmetric Primitives for MPC/FHE/ZK Applications

New applications including

- secure multi-party computation (MPC),
- fully homomorphic encryption (FHE),
- zero-knowledge proofs (ZK),

require symmetric-key primitives that

(1) *are naturally defined over* $(\mathbb{F}_p)^n$ *for a* **large** *prime integer* p (usually, $p \approx 2^{128}$ or 2^{256});

(2) *minimize their multiplicative complexity*, that is, the number of multiplications ($=$ non-linear operations) required to compute and/or verify them.

Invertible Non-Linear Operations over $\mathbb{F}_p^n$

Due to the size of p, the non-linear operations

- cannot be pre-computed and stored (no look-up tables);
- they must admit a simple algebraic expression.

Current known invertible non-linear operations:

- power map $x \mapsto x^d$ over $\mathbb{F}_p$ where $\gcd(d, p-1) = 1$;
- Dickson polynomial $x \mapsto D_{d,\alpha}(x) = \sum_{i=0}^{\lfloor d/2 \rfloor} \frac{d}{d-i} \binom{d-i}{i} \cdot (-\alpha)^i \cdot x^{d-2i}$ over $\mathbb{F}_p$ where $\gcd(d, p^2-1) = 1$;
- non-linear functions over $\mathbb{F}_p$ via Legendre function $x \mapsto L_p(x) = x^{\frac{p-1}{2}} \in \{-1, 0, 1\}$ or/and $x \mapsto (-1)^x$ operator;
- non-linear layers over $\mathbb{F}_p^n$ instantiated via Feistel and/or Lai-Massey schemes, e.g., $(x_0, x_1) \mapsto (x_1, x_1^2 + x_0)$.

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Goals

- Changing d in base of p (e.g., $\gcd(d, p-1) = 1$) is not desirable:
  - potentially harder (algebraic) security analysis which must be adapted depending on p and so on d (e.g., density of the polynomial representation);
  - efficiency could depend on the choice of d.
- Feistel and/or Lai-Massey schemes are "partially linear" (do not provide "full non-linearity").

Goal: construct new *invertible "full" non-linear layers* over $\mathbb{F}_p^n$ that

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Shift Invariant Lifting Functions $\mathcal{S}_F$ over $\mathbb{F}_p^n$ Induced by a Local Map $F : \mathbb{F}_p^m \to \mathbb{F}_p$

Let $\mathcal{S} : \mathbb{F}_p^n \to \mathbb{F}_p^n$ be a generic non-linear function:

$$\begin{aligned} \mathcal{S}(x_0, x_1, \ldots, x_{n-1}) &= y_0 \| y_1 \| \ldots \| y_{n-1} \qquad \text{where} \\ \forall i \in \{0, 1, \ldots, n-1\} : \qquad & y_i := F_i(x_0, x_1, \ldots, x_{n-1}) \end{aligned}$$

for certain $F_i : \mathbb{F}_p^n \to \mathbb{F}_p$.

$\Longrightarrow$ *Too many possible cases to analyze!*

Idea: *define $\mathcal{S}$ as a Cellular Automata (CA)*, that is, a shift-invariant transformation over a $\mathbb{F}_p^n$–array of cells defined by a *single local* update rule $F : \mathbb{F}_p^m \to \mathbb{F}_p$ for $1 \leq m \leq n$.

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SI-Lifting Functions $\mathcal{S}_F$ (2/2)

The Shift Invariant (SI) lifting function $\mathcal{S}_F : \mathbb{F}_p^n \to \mathbb{F}_p^n$ induced by $F : \mathbb{F}_p^m \to \mathbb{F}_p$ is defined as

$$\mathcal{S}_F(x_0, x_1, \ldots, x_{n-1}) = y_0 \| y_1 \| \ldots \| y_{n-1} \qquad \text{where}$$
$$\forall i \in \{0, 1, \ldots, n-1\} : \qquad y_i := F(x_i, x_{i+1}, \ldots, x_{i+m-1}) \, .$$

"Shift Invariant" property due to the fact that:

$$\Pi_i \circ \mathcal{S}_F = \mathcal{S}_F \circ \Pi_i$$

for each shift function Π_i over $\mathbb{F}_p^n$ defined as

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Example of SI-Lifting Functions over $\mathbb{F}_2^n$

See Joan Daemen's PhD Thesis ("Cipher and Hash Function Design Strategies based on linear and differential cryptanalysis"):

- given the chi function $\chi : \mathbb{F}_2^3 \to \mathbb{F}_2$:

$$\chi(x_0, x_1, x_2) = x_0 \oplus (x_1 \oplus 1) \cdot x_2 \,,$$

  then $\mathcal{S}_\chi$ over $\mathbb{F}_2^n$ is invertible if and only if $\gcd(n, 2) = 1$;

- given the function

$$F(x_0, x_1, x_2, x_3) = x_0 \oplus (x_1 \oplus 1) \cdot x_2 \cdot x_3 \,,$$

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  then $\mathcal{S}_F$ over $\mathbb{F}_2^n$ is invertible for each $n \geq 6$.

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Our Goal

Let

- $\mathbf{p \geq 3}$;
- $\boldsymbol{F : \mathbb{F}_p^m \to \mathbb{F}_p}$ **quadratic**.

Given $\mathcal{S}_F : \mathbb{F}_p^n \to \mathbb{F}_p^n$ defined as before, that is,

$$\mathcal{S}_F(x_0, x_1, \ldots, x_{n-1}) = y_0 \| y_1 \| \ldots \| y_{n-1} \qquad \text{where}$$
$$\forall i \in \{0, 1, \ldots, n-1\} : \qquad y_i := F(x_i, x_{i+1}, \ldots, x_{i+m-1}),$$

then

- is it possible to find F for which $\mathcal{S}_F$ is invertible?
- if yes, for any value of n and/or m?

SI-Lifting Functions $\mathcal{S}_F$ over $\mathbb{F}_p^n$ via Quadratic $F : \mathbb{F}_p^m \to \mathbb{F}_p$: Results for $m \in \{2, 3\}$

Necessary Conditions for Inveritibility

Let $F : \mathbb{F}_p^m \to \mathbb{F}_p$ be a quadratic function:

$$F(x_0, x_1, \ldots, x_{m-1}) := \sum_{0 \le i_0+i_1+\ldots+i_{m-1} \le 2} \alpha_{i_0,i_1,\ldots,i_{m-1}} \cdot x_0^{i_0} \cdot x_1^{i_1} \cdot \ldots \cdot x_{m-1}^{i_{m-1}} .$$

Let $\alpha^{(d)}$ be the sum of the coefficients of the degree-d monomials:

$$\alpha^{(d)} := \sum_{i_0+i_1+\ldots+i_{m-1}=d} \alpha_{i_0,i_1,\ldots,i_{m-1}} .$$

Necessary requirements for invertibility of $\mathcal{S}_F$:

$$\alpha^{(2)} = 0 \quad \text{and} \quad \alpha^{(1)} \neq 0 .$$

- If $\alpha^{(2)} = \alpha^{(1)} = 0$: $F(x, x, \ldots, x) = F(0, 0, \ldots, 0)$;
- If $\alpha^{(2)} \neq 0$: $F(x, x, \ldots, x) = \alpha^{(2)} \cdot x^2 + \alpha^{(1)} \cdot x + \alpha_{0,0,\ldots,0}$, hence collisions $\mathcal{S}_F(x', x', \ldots, x') = \mathcal{S}_F(\hat{x}, \hat{x}, \ldots, \hat{x})$.

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Main Result for $m = 2$

Theorem
Let $p \geq 3$ be a prime, let $m = 2$, and let $n \geq 2$. Let $F : \mathbb{F}_p^2 \to \mathbb{F}_p$ be a quadratic function:

$$F(x_0, x_1) = \alpha_{2,0} \cdot x_0^2 + \alpha_{1,1} \cdot x_0 \cdot x_1 + \alpha_{0,2} \cdot x_1^2 + \alpha_{1,0} \cdot x_0 + \alpha_{0,1} \cdot x_1 \,.$$

Given $\mathcal{S}_F$ over $\mathbb{F}_p^n$:

- *if $n = 2$, then $\mathcal{S}_F$ is invertible if and only if*

  $$F(x_0, x_1) = \gamma_0 \cdot x_0 + \gamma_1 \cdot x_1 + \gamma_2 \cdot (x_0 - x_1)^2$$

  for $\gamma_0 \neq \pm\gamma_1$;
- *if $n \geq 3$, then $\mathcal{S}_F$ is* **never** *invertible.*

Sketch of the Proof – Case: $m = 2$ and $n \geq 3$ (1/2)

Collisions over $\mathbb{F}_p^3$ of the form

$$\mathcal{S}_F(0, x_0, x_1) = \mathcal{S}_F(0, x_0', x_1'),$$

imply collisions over $\mathbb{F}_p^n$ for each $n \geq 3$ of the form

$$\mathcal{S}_F(0, x_0, x_1, 0, 0, \ldots, 0) = \mathcal{S}_F(0, x_0', x_1', 0, 0, \ldots, 0).$$

Indeed, both are satisfied by

$$F(0, x_0) = F(0, x_0'), \quad F(x_0, x_1) = F(x_0', x_1'), \quad F(x_1, 0) = F(x_1', 0).$$

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Necessary requirements for invertibility of $\mathcal{S}_F$:

- $\alpha_{2,0} + \alpha_{1,1} + \alpha_{0,2} = 0$;
- $\alpha_{1,0} + \alpha_{0,1} \neq 0$.

In the paper, collisions are proposed in order to cover all the cases just given. E.g., if $\alpha_{2,0}, \alpha_{1,1} \neq 0$ with $\alpha_{2,0} + \alpha_{1,1} + \alpha_{0,2} = 0$:

$$\mathcal{S}_F\left(0, \frac{\alpha_{0,2} \cdot \alpha_{1,0}}{\alpha_{1,1} \cdot \alpha_{2,0}} - \frac{\alpha_{0,1}}{\alpha_{1,1}}, x\right) = \mathcal{S}_F\left(0, \frac{\alpha_{0,2} \cdot \alpha_{1,0}}{\alpha_{1,1} \cdot \alpha_{2,0}} - \frac{\alpha_{0,1}}{\alpha_{1,1}}, -x - \frac{\alpha_{1,0}}{\alpha_{2,0}}\right)$$

for each $x \in \mathbb{F}_p$.

Examples of Invertible SI-Lifting Functions $\mathcal{S}_F$ for $m = 3$ and $n \in \{3, 4\}$

- *Case $n = m = 3$:* given
$$F(x_0, x_1, x_2) = \sum_{i=0}^{2} \mu_i \cdot x_i + (x_0 - x_1)^2 + (x_1 - x_2)^2 + (x_0 - x_2)^2 \, ,$$
such that $\text{circ}(\mu_0, \mu_1, \mu_2) \in \mathbb{F}_p^{3\times 3}$ is invertible, then $\mathcal{S}_F$ over $\mathbb{F}_p^3$ is invertible.

- *Case $n = 3$ and $m = 4$:* given
$$F(x_0, x_1, x_2) = \alpha \cdot (x_0 + x_2) + \beta \cdot x_1 + (x_0 - x_2)^2 \, ,$$
such that $\alpha \neq \pm\beta/2$, then $\mathcal{S}_F$ over $\mathbb{F}_p^4$ is invertible.

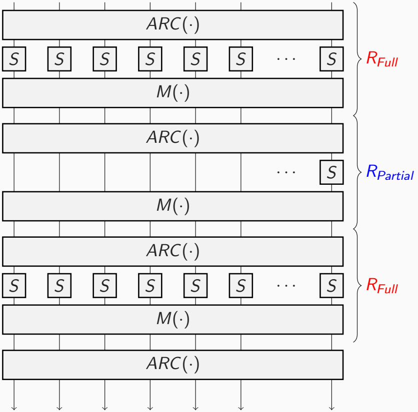
- Other examples given in the paper.

Main Result for $m = 3$ and $n \geq 5$

Theorem
Let $p \geq 3$ be a prime, let $m = 3$, and let $n \geq 5$. Let $F : \mathbb{F}_p^3 \to \mathbb{F}_p$ be **any** *quadratic function. The SI-lifting function $\mathcal{S}_F$ over $\mathbb{F}_p^n$ induced by F is* **never** *invertible.*

- Strategy of the proof similar to the one just proposed for $m = 2$ and $n \geq 3$.
- *Different from the binary case*, for which $\mathcal{S}_F$ over $\mathbb{F}_2^n$ can be invertible depending on $F : \mathbb{F}_2^3 \to \mathbb{F}_2$ and on n (e.g., χ).

The Sponge Hash Function Neptune



- $S(x) = x^d$ where $d \geq 3$ s.t. $\gcd(d, p-1) = 1$;
- Linear layer: multiplication with a MDS matrix in $\mathbb{F}_p^{t \times t}$ (that prevents infinitely long subspace trails);
- Random constants addition in $\mathbb{F}_p^t$.
- Number of rounds ($\kappa \approx \log_2(p)$):

$$R_F = 2 \cdot R_f = 8\,,$$
$$R_P \approx \log_d(p)$$

From Poseidon to Neptune

- Internal partial rounds are crucial for increasing the degree of the permutation, and so preventing algebraic attacks. Cost of

$$\underbrace{(\text{Hw}(d) + \lfloor \log_2(d) \rfloor - 1)}_{\geq 2} \cdot \underbrace{R_P}_{\approx \log_d(p)}$$

multiplications, which is *independent of t*;

- External full rounds guarantee security against statistical attacks, including differential, linear, and so on. Cost of

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Neptune's External Rounds: Non-Linear Layer

- Given any quadratic $F : \mathbb{F}_p^{\leq 3} \to \mathbb{F}_p$, then $\mathcal{S}_F$ over $\mathbb{F}_p^{\geq 5}$ is **not** invertible.
- Let $t = 2 \cdot t'$ even. Non-linear layer of NEPTUNE's external rounds via concatenation of S-Boxes $\mathcal{S}$ over $\mathbb{F}_p^2$, defined as

$$\mathcal{S}(x_0, x_1) = \mathcal{S}' \circ \mathcal{A} \circ \mathcal{S}'(x_0, x_1)$$

where (for $\gamma \neq 0$):

$$\mathcal{S}'(x_0, x_1) = x_0 + (x_0 - x_1)^2 \| x_1 + (x_0 - x_1)^2 ;$$

$$\mathcal{A}(x_0, x_1) = \begin{bmatrix} \gamma \\ 0 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \times \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} ;$$

- Differential property of $\mathcal{S}$: $\mathrm{DP}_{\max} = p^{-1}$;
- Cost of t *multiplications for computing* $\mathcal{S}$ (versus $\geq 2 \cdot t$ for power maps).

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Neptune versus Poseidon – $S(x) = x^5$

Table: Comparison of Poseidon and Neptune – both instantiated with $d = 5$ – for the case $p \approx 2^{128}$ (or bigger), $\kappa = 128$, and several values of $t \in \{4, 8, 12, 16\}$.

	t	R_F	R_P & R_I	Multiplicative Complexity
Poseidon ($d = 5$)	4	8	60	276 (+ 21.0 %)
Neptune ($d = 5$)	4	6	68	**228**
Poseidon ($d = 5$)	8	8	60	372 (+ 40.1 %)
Neptune ($d = 5$)	8	6	72	**264**
Poseidon ($d = 5$)	12	8	61	471 (+ 53.9 %)
Neptune ($d = 5$)	12	6	78	**306**
Poseidon ($d = 5$)	16	8	61	567 (+ 64.3 %)
Neptune ($d = 5$)	16	6	83	**345**

(See the paper for more details about Neptune' specification.)

Summary and Open Problems

- Let $p \geq 3$. Given any quadratic function $F : \mathbb{F}_p^m \to \mathbb{F}_p$, then the SI-lifting function $\mathcal{S}_F$ over $\mathbb{F}_p^n$ is **not** invertible if
  - $m = 1$, $n \geq 1$;
  - $m = 2$, $n \geq 3$;
  - $m = 3$, $n \geq 5$.
- **Open Conjecture:** Given F as before, $\mathcal{S}_F$ is **never** invertible if $n \geq 2 \cdot m - 1$;
- **Open Problem:** Construct invertible non-linear functions over $\mathbb{F}_p^n$ with minimal multiplicative complexity;
- Exploit them when designing future MPC-/ZK-/FHE-friendly symmetric schemes!

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Thanks for your attention!

Questions?

Comments?

Lai-Massey Schemes: Example of SI-Lifting Functions for $m = n$

Let $\text{circ}(\mu_0, \mu_1, \ldots, \mu_{n-1}) \in \mathbb{F}_p^{n\times n}$ be an invertible circulant matrix. Given an invertible even function $H : \mathbb{F}_p \to \mathbb{F}_p$ (i.e., $H(z) = H(-z)$), let

$$F(x_0, x_1, \ldots, x_{n-1}) = \sum_{i=0}^{n-1} \mu_i \cdot x_i + H\left(\sum_{i=0}^{n-1} (-1)^i \cdot x_i\right) .$$

If $n = 2n'$ is even, then $\mathcal{S}_F$ over $\mathbb{F}_p^n$ is invertible.

Proof. Given $\mathcal{S}_F(x_0, x_1, \ldots, x_{n-1}) = y_0 \| y_1 \| \ldots \| y_{n-1}$:

- if $\text{circ}(\mu_0, \mu_1, \ldots, \mu_{n-1}) = \text{circ}(1, 0, \ldots, 0)$, then $\sum_{i=0}^{n-1} (-1)^i \cdot x_i = \sum_{i=0}^{n-1} (-1)^i \cdot y_i$;
- otherwise, work with $z \in \mathbb{F}_p^n$ defined as $z = \text{circ}^{-1}(\mu_0, \mu_1, \ldots, \mu_{n-1}) \times y$.

(Other examples in the paper.)

Another Necessary Conditions for Inveritibility

Definition. A function $F : \mathbb{F}_p^m \to \mathbb{F}_p$ is balanced if and only if

$$\forall y \in \mathbb{F}_p : \qquad \left|\{x \in \mathbb{F}_p^m \mid F(x) = y\}\right| = p^{m-1} \, .$$

Lemma. If F is not balanced, then $\mathcal{S}_F$ is **not** invertible.

Example. Let $p \geq 2$ be a prime, and let $F : \mathbb{F}_p^2 \to \mathbb{F}_p$ be

$$F(x_0, x_1) = \alpha_{2,0} \cdot x_0^2 + \alpha_{1,1} \cdot x_0 \cdot x_1 + \alpha_{0,2} \cdot x_1^2 + \alpha_{1,0} \cdot x_0 + \alpha_{0,1} \cdot x_1 \, .$$

If $\alpha_{2,0} = \alpha_{0,2} = 0$, then F is **not** a balanced function.

Neptune's External Rounds: Linear Layer

Given $M', M'' \in \mathbb{F}_p^{t' \times t'}$ two MDS matrices, linear layer $M \in \mathbb{F}_p^{t \times t}$ of NEPTUNE's external rounds defined as

$$M_{i,j} = \begin{cases} M'_{i',j'} & \text{if } (i,j) = (2i', 2j') \\ M''_{i'',j''} & \text{if } (i,j) = (2i''+1, 2j''+1) \\ 0 & \text{otherwise} \end{cases},$$

that is,

$$M = \begin{bmatrix} M'_{0,0} & 0 & M'_{0,1} & 0 & \dots & M'_{0,t'-1} & 0 \\ 0 & M''_{0,0} & 0 & M''_{0,1} & \dots & 0 & M''_{0,t'-1} \\ M'_{1,0} & 0 & M'_{1,1} & 0 & \dots & M'_{1,t'-1} & 0 \\ 0 & M''_{1,0} & 0 & M''_{1,1} & \dots & 0 & M''_{1,t'-1} \\ \vdots & & & & \ddots & & \vdots \\ M'_{t'-1,0} & 0 & M'_{t'-1,1} & 0 & \dots & M'_{t'-1,t'-1} & 0 \\ 0 & M''_{t'-1,0} & 0 & M''_{t'-1,1} & \dots & 0 & M''_{t'-1,t'-1} \end{bmatrix}.$$

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