Revisiting and Improving Algorithms for the 3XOR Problem

Charles Bouillaguet[1] Claire Delaplace[1,2] Pierre-Alain Fouque[2]

[2] University of Rennes 1, IRISA, France

[1] University of Lille, CRIStAL, France

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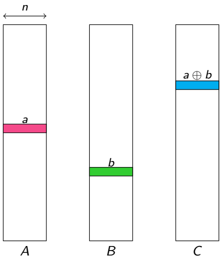
3XOR Problem

Problem

Given three lists A, B, and C of uniformly random elements of $\{0,1\}^n$, find $(a,b,c) \in A \times B \times C$, such that $a \oplus b \oplus c = 0$.



- Difficult case of Generalised Birthday Problem
- Application in cryptanalysis of some authenticated encryption scheme
- Lists formed by querying oracles $\Rightarrow$ can be as big as we want
- $|A| \cdot |B| \cdot |C| \geq 2^n \Rightarrow$ solution w.h.p.

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- Space: $\mathcal{O}\left(|A| + |B| + |C|\right)$
- $|A| = |B| = |C| = 2^{n/3} \Rightarrow$ Time: $\mathcal{O}\left(2^{2n/3}\right)$, Space: $\mathcal{O}\left(2^{n/3}\right)$
- $|A| = |B| = 2^{n/4}$, $|C| = 2^{n/2} \Rightarrow$ Time: $\mathcal{O}\left(2^{n/2}\right)$, Space: $\mathcal{O}\left(2^{n/2}\right)$

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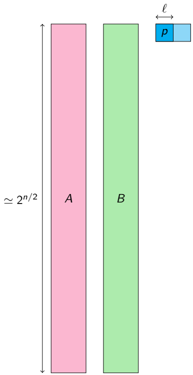
Time/Space tradeoff: Well studied in the past (e.g. [Wagner02], [Bernstein07]).

Wagner and its descendants



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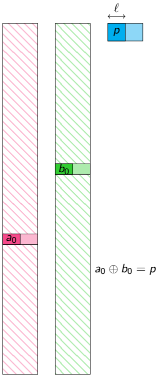

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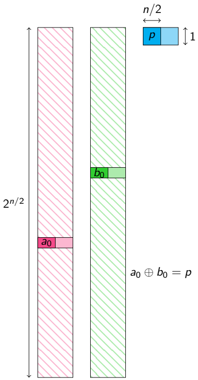

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[Wagner02]: $2^{n/2}$ queries allowed

$|C| = 1$.

Time/Space $\mathcal{O}\left(2^{n/2}\right)$

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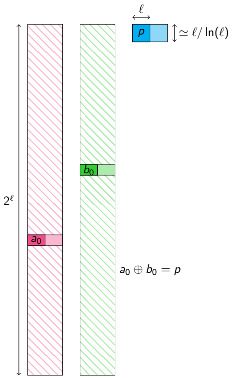

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[NS14]: $2^\ell \simeq \frac{2^{n/2}}{\sqrt{(n/2)/\ln(n/2)}}$ queries allowed

p: Most frequent prefix in C

Time/Space $\mathcal{O}\left(2^{n/2}/\sqrt{n/\ln(n)}\right)$

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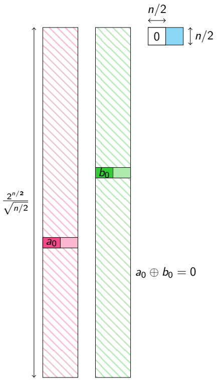

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[Joux09]: $2^{n/2}/\sqrt{n/2}$ queries allowed

$|C| = n/2$, Basis change to force $p = 0$

Time/Space $\mathcal{O}\left(2^{n/2}/\sqrt{n}\right)$

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96-bit 3XOR

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$\Rightarrow$ Keep the lists small!

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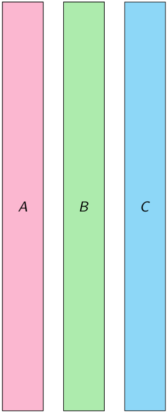
$2^{n/2}$ Queries

- $\ell = n/4$, $n' = 3n/4$
- **Stored data:** $\mathcal{O}\left(2^{n/4}\right)$ words
- **Time Complexity:** $\mathcal{O}\left(2^{n/2}\right)$ with Quadratic Algorithm

Our Work: A Generalization of Joux Algorithm



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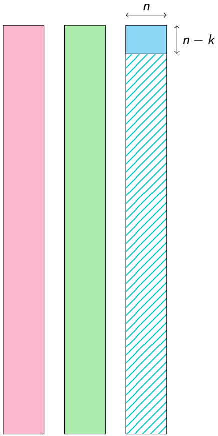

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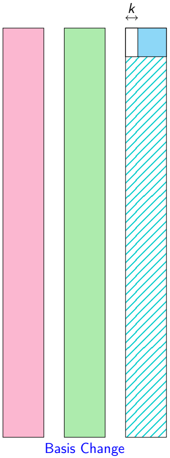

Generalization to any size of input lists

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- Apply Joux's Algorithm

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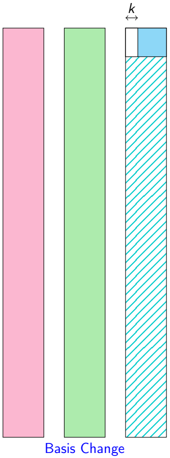

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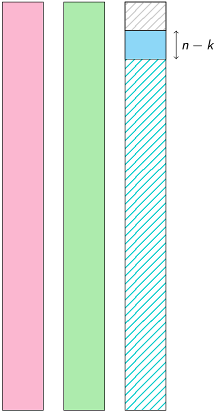

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- Re-iterate with $n-k$ other rows...

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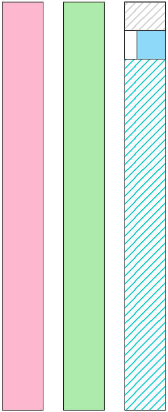

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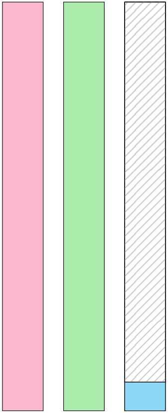

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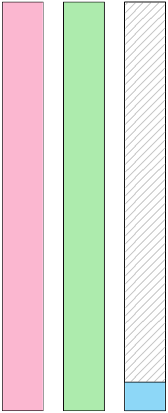

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$$k = \log_2(\min(|A|, |B|)), \text{ Time: } \mathcal{O}\left((|A| + |B|) \cdot \frac{|C|}{n}\right)$$

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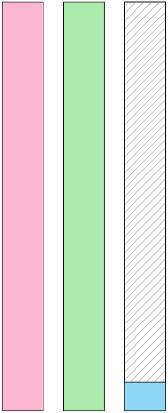

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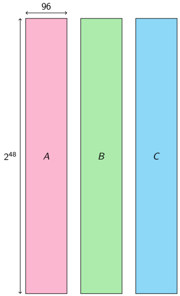
$$|A| = |B| = |C| = 2^{n/3};\, k = n/3, \text{ Time: } \mathcal{O}\left(\frac{2^{2n/3}}{n}\right)$$

A Concrete Example

A 96-bit 3XOR



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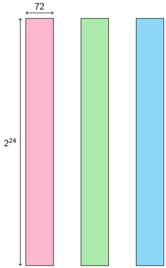

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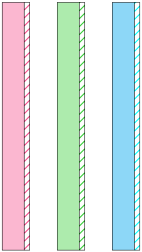

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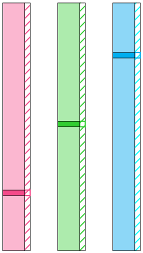

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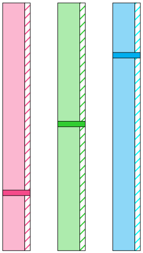

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Experimentations

- 3XOR of 96 bits of SHA-256
- Tests performed on a Haswell Core i5 CPU

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CPU hours	340	105
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Creation of the lists: $\times 100$ slower than processing them!

In a Nutshell

This Algorithm...

- can be applied to any size of input list
- has a $\times n$ speed-up compared to the Quadratic Algorithm
- is about 3 times faster, in practice ($n = 96$)
- is faster than [NS14] with the same amount of data, in theory
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Possible improvements

Find basis changes that increase the size of the sublists

- We propose two ways of doing this
- Only a constant time improvement in theory

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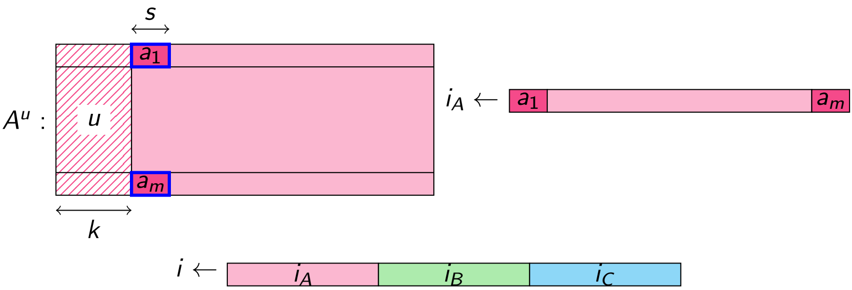
  - Test s-bit partial collision with a hash table
- If the test fail: no solution for sure
- If the test succeed: there may be a solution
  - Solve the small instance

Preliminary Test

Instance $(A^u, B^v, C^{u \oplus v})$



$$T[i] = 1 \iff \exists j, k, \ell \text{ s.t. } a_j \oplus b_k \oplus c_\ell = 0$$

$$T[i] = 0 \Rightarrow \text{No solution in } (A^u, B^v, C^{u \oplus v})$$

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BDP In Theory

When n grows up to infinity, only one triplet passes the test
$\implies$ complexity of the algorithm:

$$\text{Time: } \mathcal{O}\left(\frac{2^{2n/3}\log^2(n)}{n^2}\right), \text{Space: } \mathcal{O}\left(2^{n/3}\right)$$

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BDP In Practice

$n = 96$, **machine words:** 64 **bits**
Expected size of a bucket: $m = 0.14$

$\implies$ Completely impractical

Conclusion

This work

- Discusses issues arising from the 3XOR problem
- Propose a new practical algorithm for the 3XOR problem, that is
  - $n\times$ faster than the quadratic algorithm in theory
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What's Next?

- Compute a 128-bit 3XOR on SHA-256
- Expect to have the lists in about 2 years (using one Antminer S7)

Code available here:

`https://github.com/cbouilla/3XOR`

Thank you for your time!