Multivariate Profiling of Hulls for Linear Cryptanalysis

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March 7, 2018

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Linear Cryptanalysis in a Nutshell

- Introduced in 1993 by Matsui to analyse DES.
- Uses linear approximations (α, β) over E_K with large absolute correlation defined by

$$C^K_{\alpha,\beta} = 2 \cdot \Pr(\langle \alpha, x \rangle \oplus \langle \beta, E_K(x) \rangle = 0) - 1,$$

as a distinguisher.

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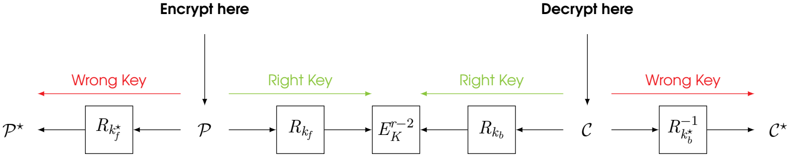

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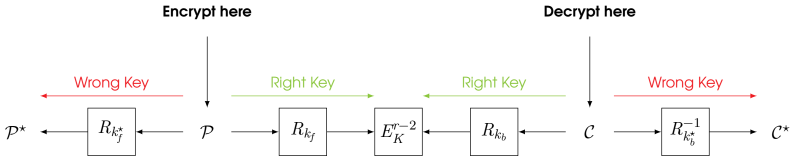

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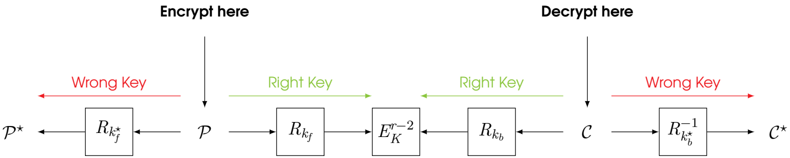

- High absolute correlation $\longrightarrow$ Likely right key guess.
- Low absolute correlation $\longrightarrow$ Likely wrong key guess.
- Central question when estimating attack complexity:

How is the correlation distributed for right and wrong key guesses?

Key Dependent Correlations

- The key-dependent correlation can be expressed as (Daemen, Rijmen, 2007)

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- $|C_{\alpha,\beta}^{K}|$ is not constant for ciphers with a strong linear hull effect.

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- Difficult to analyse if the multivariate distributions of the C_i^K or η_i^K are not "simple".

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- Most ciphers do not have independent round-keys.
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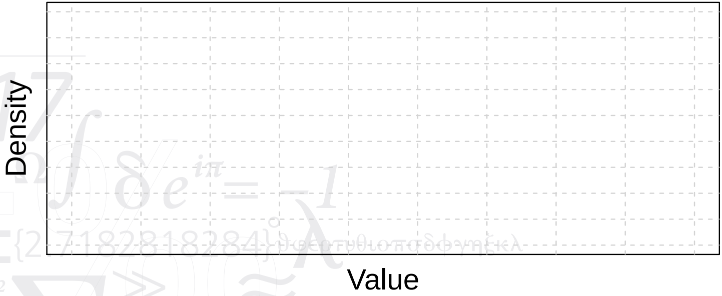
$\boxed{\text{We propose Multivariate Linear Cryptanalysis as a next step}}$

Multivariate Linear Cryptanalysis – Three Steps

The Main Model: Arbitrary Right-Key Distribution

- We consider a set of M linearly independent approximations.



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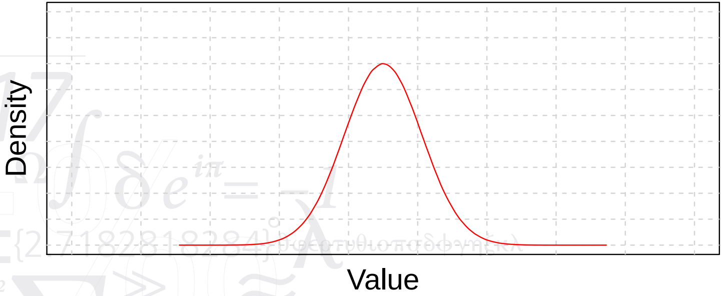

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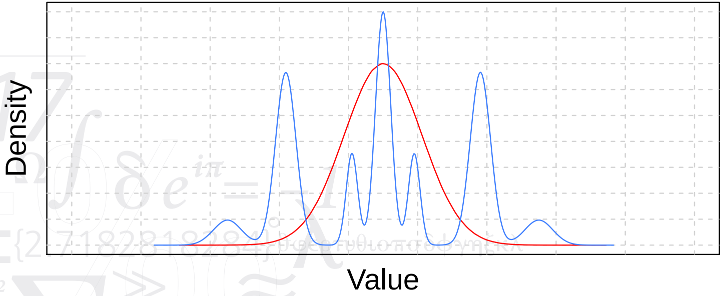

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- The rest of the hull is modeled as noise:

$$\mathcal{N}(0, 2^{-n}).$$

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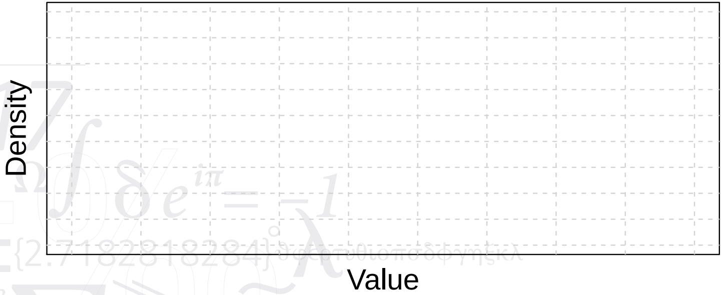
The Practical Model: Profiling with Signal/Noise Decomposition



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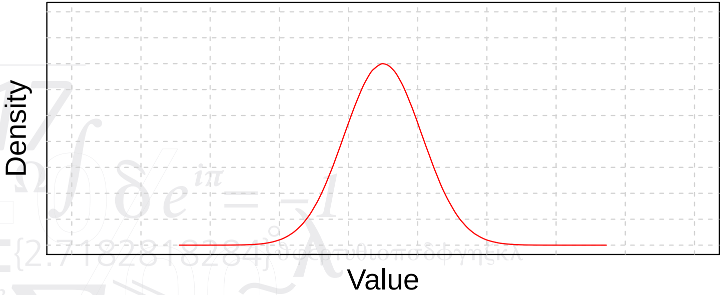

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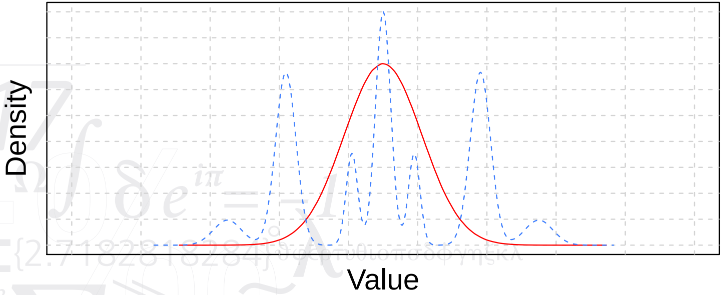

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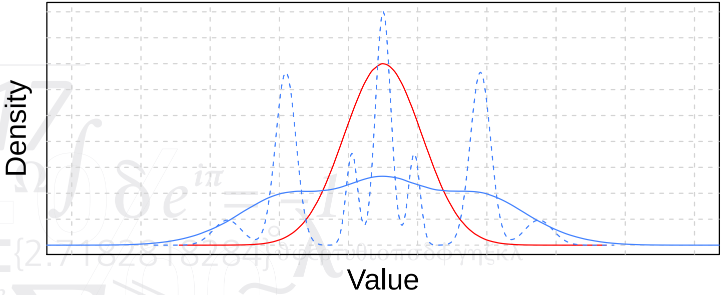

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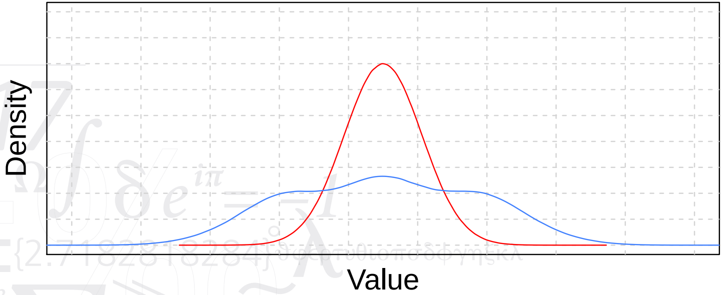

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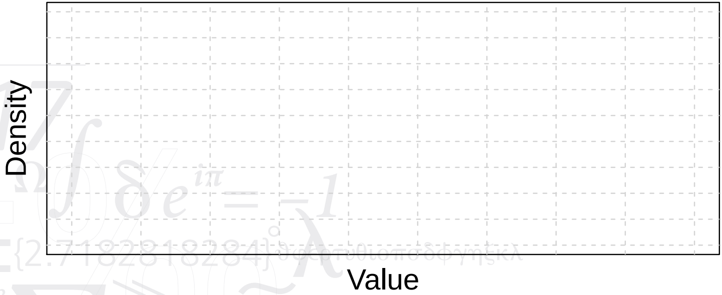

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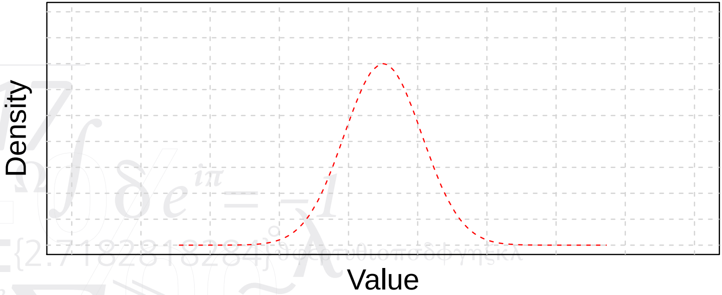

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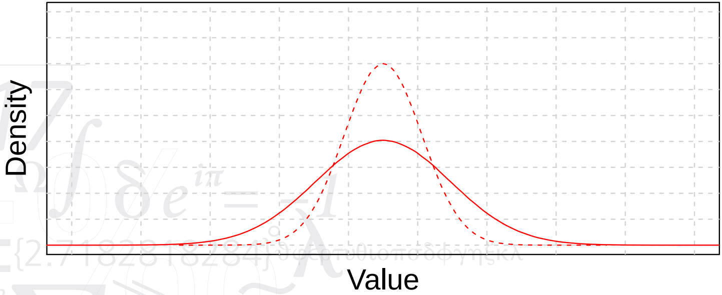

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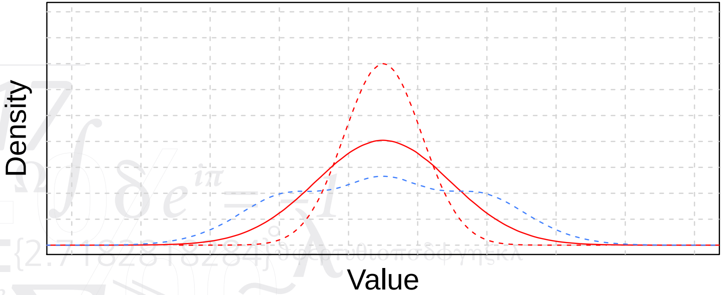

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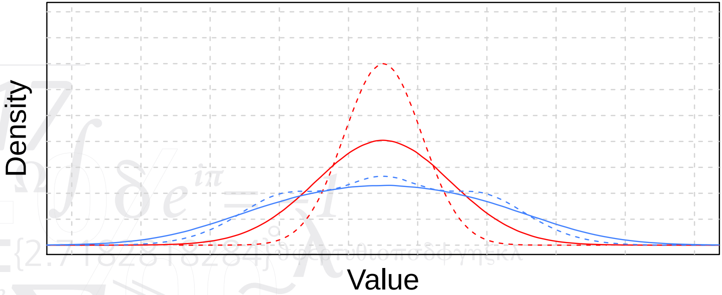

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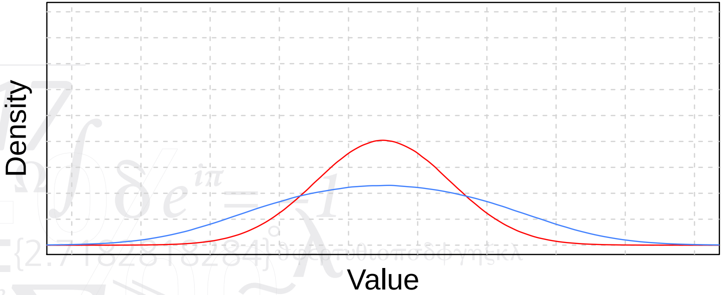

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Blondeau and Nyberg (2017)	✓	✓	✓	✓	✗	✗	✓	?
Blondeau and Nyberg (2016)	✓	✓	✓	✓	✓	✗	✗	✓*
This work	✓	✓	✓	✓	✓	✓	✓	✓*

An Application: New Attacks on PRESENT

- We consider a set of 135 linearly independent approximations over 22/23 rounds of PRESENT.
- We used a method similar to the partial, sparse correlation matrix method by Abdelraheem (2012) to compute $\mathcal{D}^{\star}_{135}$.
- The low number of approximations allow for efficient key-guessing over 4 rounds.

An Application: New Attacks on PRESENT

Rounds	Success probability	#Approximations	Time complexity	Data complexity	Memory complexity	Key-dependent	Reference
25	95%	2295	$2^{65.0}$	$2^{62.4}$	$2^{34.0}$		Cho (2010)
	95%	2295	$2^{65.0}$	$2^{61.6}$	$2^{34.0}$	✓	Huang *et al.* (2015)
	74%	2295	$2^{72.0}$	$2^{61.0}$	$2^{34.0}$	✓	Blondeau and Nyberg (2016)
26	95%	2295	$2^{72.0}$	$2^{64.0}$	$2^{34.0}$		Cho (2010)
	80%	2295	$2^{76.0}$	$2^{62.5}$	$2^{34.0}$	✓	Huang *et al.* (2015)
	51%	2295	$2^{72.0}$	$2^{63.8}$	$2^{34.0}$	✓	Blondeau and Nyberg (2016)
	95%	**135**	$\mathbf{2^{68.6}}$	$\mathbf{2^{63.0}}$	$\mathbf{2^{48.0}}$	✓	**This work**
27	95%	405	$2^{74.0}$	$2^{64.0}$	$2^{70.0}$		Zheng and Zhang (2015)
	95%	**135**	$\mathbf{2^{77.3}}$	$\mathbf{2^{63.8}}$	$\mathbf{2^{48.0}}$	✓	**This work**

Thank you

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