Shorter Linear Straight-Line Programs for MDS Matrices

Yet another XOR Count Paper

Thorsten Kranz[1], Gregor Leander[1], Ko Stoffelen[2], Friedrich Wiemer[1]

[1]Horst Görtz Institute for IT Security, Ruhr-Universität Bochum, Germany

[2]Digital Security Group, Radboud University, Nijmegen, The Netherlands

Lightweight Cryptography

Cryptographic systems might have to fulfill special constraints.

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Cryptographic systems might have to fulfill special constraints.

Typical Goal

Minimize the chip-area.

Linear Layers

- Matrix multiplication(s).
- Often MDS matrices.

$$\begin{pmatrix} 02 & 03 & 01 & 01 \\ 01 & 02 & 03 & 01 \\ 01 & 01 & 02 & 03 \\ 03 & 01 & 01 & 02 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix}, \quad x_i, y_i \in \mathbb{F}_{2^8}$$

Goal: Small round-based implementation

$$\begin{pmatrix} \texttt{02} & \texttt{03} & \texttt{01} & \texttt{01} \\ \texttt{01} & \texttt{02} & \texttt{03} & \texttt{01} \\ \texttt{01} & \texttt{01} & \texttt{02} & \texttt{03} \\ \texttt{03} & \texttt{01} & \texttt{01} & \texttt{02} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix}, \qquad x_i, y_i \in \mathbb{F}_{2^8}$$

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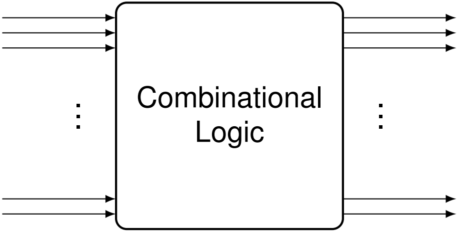

Metric: XOR count

- Implement matrix multiplication only with XOR operations.
- Use as few XORs as possible.
- Idea: Low XOR count $=$ Low chip-area

- Note: No intermediate result needs to be recomputed.

Outline

Outline

Previous Work

- **FSE 2018:** Jean, Peyrin, Sim, Tourteaux
  Optimizing Implementations of Lightweight Building Blocks
- **FSE 2017:** C. Li and Q. Wang
  Design of Lightweight Linear Diffusion Layers from Near-MDS Matrices
- **FSE 2017:** Sarkar and Syed
  Lightweight Diffusion Layer: Importance of Toeplitz Matrices
- **CRYPTO 2016:** Beierle, Kranz, Leander
  Lightweight Multiplication in $GF(2^n)$ with Applications to MDS Matrices
- **FSE 2016:** Liu and Sim
  Lightweight MDS Generalized Circulant Matrices
- **FSE 2016:** Y. Li and M. Wang
  On the Construction of Lightweight Circulant Involutory MDS Matrices
- **FSE 2015:** Sim, Khoo, Oggier, Peyrin
  Lightweight MDS Involution Matrices

Previous Work

- Searching many matrices.

Previous Work

- Searching many matrices.
  Cauchy

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- Searching many matrices.
  Cauchy, *Vandermonde*

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- Searching many matrices.
  Cauchy, *Vandermonde*, *Circulant*

Previous Work

- Searching many matrices.
  Cauchy, *Vandermonde*, *Circulant*, *Hadamard*

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- Searching many matrices.
  Cauchy, *Vandermonde*, *Circulant*, *Hadamard*, *Hadamard-Cauchy*

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- Searching many matrices.
  Cauchy, *Vandermonde*, *Circulant*, *Hadamard*, *Hadamard-Cauchy*, *Toeplitz*

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- Searching many matrices.
  Cauchy, *Vandermonde*, *Circulant*, *Hadamard*, *Hadamard-Cauchy*, *Toeplitz*, *Arbitrary*

Previous Work

- Searching many matrices.
  Cauchy, *Vandermonde*, *Circulant*, *Hadamard*, *Hadamard-Cauchy*, *Toeplitz*, *Arbitrary*
- Optimizing element multiplication.

Counting XORs: Overhead and Fixed Cost

- The XOR count is typically split into overhead and fixed cost.

Matrix Multiplication

$$\begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} & \cdots & \alpha_{1,n} \\ \alpha_{2,1} & & & \\ \vdots & & \ddots & \\ \alpha_{n,1} & & & \alpha_{n,n} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}, \quad \alpha_{i,j}, x_i, y_i \in \mathbb{F}_{2^k}$$

$$\underbrace{\sum_{i,j} \mathrm{XOR}(\alpha_{i,j})}_{\text{Overhead}} + \underbrace{n \cdot (n-1) \cdot k}_{\text{Fixed Cost}}$$

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Previous Results

Table: Best XOR counts of previous work. Matrices in the lower half are involutory.

Dimension	S-box	XOR count
4×4	4 bit	$10 + 48$
4×4	8 bit	$10 + 96$
8×8	4 bit	$160 + 224$
8×8	8 bit	$192 + 448$
4×4	4 bit	$15 + 48$
4×4	8 bit	$30 + 96$
8×8	4 bit	$200 + 224$
8×8	8 bit	$288 + 448$

Previous Results

Table: Best XOR counts of previous work. Matrices in the lower half are involutory.

Dimension	S-box	XOR count
4×4	4 bit	58
4×4	8 bit	106
8×8	4 bit	384
8×8	8 bit	640
4×4	4 bit	63
4×4	8 bit	126
8×8	4 bit	424
8×8	8 bit	736

Outline

Local Optimization

Optimize $k \times k$ matrix over $\mathbb{F}_2$.

$$M = \begin{pmatrix} \boxed{\alpha_{1,1}} & \alpha_{1,2} & \cdots & \alpha_{1,n} \\ \alpha_{2,1} & & & \\ \vdots & & \ddots & \\ \alpha_{n,1} & & & \alpha_{n,n} \end{pmatrix}, \quad \alpha_{i,j} \in \mathbb{F}_{2^k}$$

Global Optimization

Optimize $nk \times nk$ matrix over $\mathbb{F}_2$.

$$M = \begin{pmatrix} \boxed{\begin{matrix} \alpha_{1,1} & \alpha_{1,2} & \cdots & \alpha_{1,n} \\ \alpha_{2,1} & & & \\ \vdots & & \ddots & \\ \alpha_{n,1} & & & \alpha_{n,n} \end{matrix}} \end{pmatrix}, \quad \alpha_{i,j} \in \mathbb{F}_{2^k}$$

Global Optimization

- **BFA 2017:** Boyar, Find, Peralta
  Low-Depth, Low-Size Circuits for Cryptographic Applications
- **ePrint 2017:** Visconti, Schiavo, Peralta
  Improved upper bounds for the expected circuit complexity of dense systems of linear equations over GF(2)
- **JoC 2013:** Boyar, Matthews, Peralta
  Logic Minimization Techniques with Applications to Cryptology
- **SAT 2010:** Fuhs, Schneider-Kamp
  Synthesizing Shortest Linear Straight-Line Programs over GF(2) Using SAT
- **IWIL 2010:** Fuhs, Schneider-Kamp
  Optimizing the AES S-Box using SAT
- **MFCS 2008:** Boyar, Matthews, Peralta
  On the Shortest Linear Straight-Line Program for Computing Linear Forms
- **ISIT 1997:** Paar
  Optimized Arithmetic for Reed-Solomon Encoders

Global Optimization

- Lots of work about implementing binary matrices with few XORs.
- Goal: Find Shortest Linear Straight-Line Programs.

Global Optimization

- Lots of work about implementing binary matrices with few XORs.
- Goal: Find Shortest Linear Straight-Line Programs.
- Equivalent to our goal!
  (Hardware implementation with lowest XOR count.)

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Algorithm 1 (Paar 1997)

- Find most common subexpression.
- Add according computation to the program.

Example

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_0 + a_2 + a_3 \\ a_0 + a_1 + a_2 \\ a_0 + a_1 + a_2 + a_3 \\ a_1 + a_2 + a_3 \end{pmatrix}$$

Algorithm 1 (Paar 1997)

Example

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} {\color{red}a_0} + {\color{red}a_2} + a_3 \\ {\color{red}a_0} + a_1 + {\color{red}a_2} \\ {\color{red}a_0} + a_1 + {\color{red}a_2} + a_3 \\ a_1 + a_2 + a_3 \end{pmatrix}$$

Algorithm 1 (Paar 1997)

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$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} (a_0 + a_2) + a_3 \\ (a_0 + a_2) + a_1 \\ (a_0 + a_2) + a_1 + a_3 \\ a_1 + a_2 + a_3 \end{pmatrix}$$

$$x_0 = a_0 + a_2$$

Algorithm 1 (Paar 1997)

Example

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} x_0 + a_3 \\ x_0 + a_1 \\ x_0 + a_1 + a_3 \\ a_1 + a_2 + a_3 \end{pmatrix}$$

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$$\begin{aligned} x_0 &= a_0 + a_2 \\ x_1 &= x_0 + a_1 \\ x_2 &= a_1 + a_2 \\ x_3 &= x_0 + a_3 \\ x_4 &= x_1 + a_3 \\ x_5 &= x_2 + a_3 \end{aligned}$$

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$$\begin{aligned} x_0 &= a_0 + a_2 \\ x_1 &= x_0 + a_1 = a_0 + a_1 + a_2 \\ x_2 &= a_1 + a_2 \\ x_3 &= x_0 + a_3 = a_0 + a_2 + a_3 \\ x_4 &= x_1 + a_3 = a_0 + a_1 + a_2 + a_3 \\ x_5 &= x_2 + a_3 = a_1 + a_2 + a_3 \end{aligned}$$

Algorithm 1 (Paar 1997)

Table: New XOR counts for matrices from previous work. Matrices in the lower half are involutory.

Dimension	S-box	Previously best	New results
4×4	4 bit	58	46
4×4	8 bit	106	102
8×8	4 bit	384	210
8×8	8 bit	640	464
4×4	4 bit	63	51
4×4	8 bit	126	102
8×8	4 bit	424	222
8×8	8 bit	736	620

More advanced heuristics

- There exists many follow-up work.
- More sophisticated algorithms.

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$x_0 = a_0 + a_1$

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Outline

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  - matrices known from block ciphers and hash functions

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- We applied the heuristics to
  - matrices from previous work
  - matrices known from block ciphers and hash functions
- Could always find improved implementations (lower XOR count).

Improved Implementations

- We applied the heuristics to
  - matrices from previous work
  - matrices known from block ciphers and hash functions
- Could always find improved implementations (lower XOR count).
- Including AES MixColumns implementation with 97 XORs. (So far 103 was best.)

Statistical Analysis

- Analyzed different constructions
  Cauchy, Circulant, Hadamard, Toeplitz, Vandermonde, Arbitrary

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Good strategy

Using subfield construction with best results from smaller S-box size.

New Results

Table: New best XOR counts compared to previous work. Matrices in the lower half are involutory.

Dimension	S-box	Previously best	New best
4×4	4 bit	58	36
4×4	8 bit	106	72
8×8	4 bit	384	196
8×8	8 bit	640	392
4×4	4 bit	63	42
4×4	8 bit	126	84
8×8	4 bit	424	212
8×8	8 bit	736	424

New Results

Table: New best XOR counts compared to previous work. Matrices in the lower half are involutory.

Dimension	S-box	Previously best	New best
4×4	4 bit	$10 + 48$	$-12 + 48$
4×4	8 bit	$10 + 96$	$-24 + 96$
8×8	4 bit	$160 + 224$	$-28 + 224$
8×8	8 bit	$192 + 448$	$-56 + 448$
4×4	4 bit	$15 + 48$	$-6 + 48$
4×4	8 bit	$30 + 96$	$-12 + 96$
8×8	4 bit	$200 + 224$	$-12 + 224$
8×8	8 bit	$288 + 448$	$-24 + 448$

Conclusion

Take Home Messages

- Optimize globally rather than locally.
- Stop thinking in *overhead* and *fixed cost*.
- Use the existing heuristics.
- Not necessary to restrict to matrices over finite fields.

https://github.com/pfasante/shorter_linear_slps_for_mds_matrices

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Any Questions?