Nonlinear Approximations in Cryptanalysis Revisited

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Introduction

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Our Contribution

We study nonlinear approximations using the framework of linear cryptanalysis.

Outline

1. Our framework for (non-)linear approximations
2. Invariants imply highly-biased linear approximations (in many cases)
3. Probabilistic nonlinear approximations for cryptanalysis

(Non-)linear Approximations

- Let $F\colon \mathbb{F}_2^m \to \mathbb{F}_2^n$ be a function (e.g., a block cipher with a fixed key)
- Approximate a Boolean function h in the output by a Boolean function g in the input
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Definition: Correlation of an Approximation

Let $g : \mathbb{F}_2^m \to \mathbb{F}_2, h\colon \mathbb{F}_2^n \to \mathbb{F}_2$ be Boolean functions. The correlation of the approximation $g(x) \approx h(F(x))$ is defined as

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Example: For $\gamma \in \mathbb{F}_2^n$, let ℓ_γ be the linear function defined by

$$\ell_\gamma\colon \mathbb{F}_2^n \to \mathbb{F}_2, x \mapsto \langle \gamma, x \rangle .$$

Linear cryptanalysis exploits the existence of $\gamma, \gamma' \in \mathbb{F}_2^n$ for which $|\,\mathrm{cor}_{E_k}(\ell_\gamma, \ell_{\gamma'})| \gg 2^{-\frac{n}{2}}$.

(Nonlinear) Invariant Attacks [Todo, Leander, Sasaki 2016]

Definition: Invariant Set

Let $F : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^n$ be a permutation. $\mathcal{S} \subseteq \mathbb{F}_2^n$ is an invariant set for F if $F(\mathcal{S}) = \mathcal{S}$ or $F(\mathcal{S}) = \mathbb{F}_2^n \setminus \mathcal{S}$.

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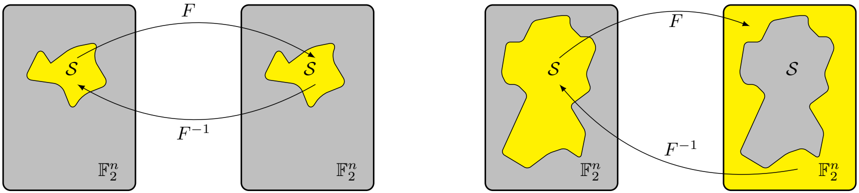

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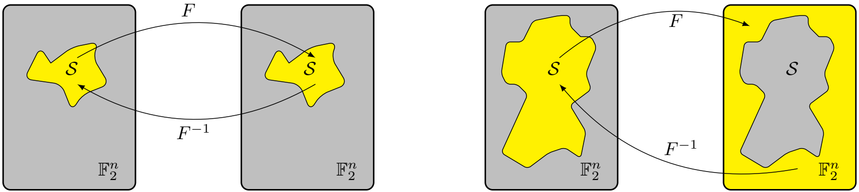

Equivalently:

Let g be the n-bit Boolean function defined by $g(x) := 1$ iff $x \in \mathcal{S}$. Then,

$$\forall x \in \mathbb{F}_2^n : g(F(x)) = g(x) \text{ or } \forall x \in \mathbb{F}_2^n : g(F(x)) = g(x) + 1.$$

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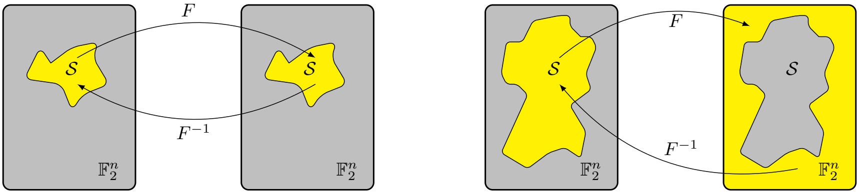

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Correlation of an invariant

$$\mathrm{cor}_F(g, g) \in \{\pm 1\}$$

Linear vs Nonlinear Approximations: Trail Composition

Thm: Linear Trail Composition [Daemen, Govaerts, Vandewalle 1995]

Let $F \colon \mathbb{F}_2^n \to \mathbb{F}_2^n$ be of the form $F = F_t \circ \cdots \circ F_1$ with $F_i \colon \mathbb{F}_2^n \to \mathbb{F}_2^n$. The correlation of an approximation $\ell_{\alpha_0}(x) \approx \ell_{\alpha_t}(F(x))$ can be given as

$$\mathrm{cor}_F(\ell_{\alpha_0}, \ell_{\alpha_t}) = \sum_{\alpha_1,\dots,\alpha_{t-1} \in \mathbb{F}_2^n} \prod_{i=1}^{t} \mathrm{cor}_{F_i}(\ell_{\alpha_{i-1}}, \ell_{\alpha_i}) \; .$$

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Let $F\colon \mathbb{F}_2^n \to \mathbb{F}_2^n$ and let $g, h\colon \mathbb{F}_2^n \to \mathbb{F}_2$. Then,

$$\mathrm{cor}_F(g, h) = \sum_{\gamma,\gamma'\in\mathbb{F}_2^n} \mathrm{cor}_g(\ell_\gamma)\, \mathrm{cor}_F(\ell_\gamma, \ell_{\gamma'})\, \mathrm{cor}_h(\ell_{\gamma'}) \; ,$$

where $\mathrm{cor}_g(\ell_\gamma) \coloneqq \mathrm{cor}_g(\ell_\gamma, \ell_1) = 2\,\mathrm{Prob}_x(\langle\gamma, x\rangle = g(x)) - 1$.

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where $\Gamma_g := \{\gamma |\, \mathrm{cor}_g(\ell_\gamma) \neq 0\}$.

The case of balanced plateaued functions (BPF)

BPF: A balanced Boolean function g such that, $\forall\gamma$: $cor_g(\ell_\gamma) \in \{0, \pm L\}$

Thm: Existence of Highly-Biased Linear Approximations (1)

Let g be a BPF which is invariant for a permutation $F\colon \mathbb{F}_2^n \to \mathbb{F}_2^n$. Then, there exists an n-bit Boolean function f such that

$$|\sum_{\gamma,\gamma' \in \Gamma_g} (-1)^{f(\gamma)+f(\gamma')} \operatorname{cor}_F(\ell_\gamma, \ell_{\gamma'})| = |\Gamma_g|$$

Moreover, there exist nonzero γ, γ' such that $|\operatorname{cor}_F(\ell_\gamma, \ell_{\gamma'})| \geq \frac{1}{|\Gamma_g|}$.

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If g is a quadratic Boolean function, then

$$\operatorname{cor}_g(\ell_\gamma) \in \{0, \pm 2^{\frac{\dim \mathrm{LS}(g) - n}{2}}\} .$$

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- Since g is invariant for each of the rounds, the existence of this linear approximation is independent on the number of rounds!

The case of invariant subspaces

Invariant subspace attack: g is the indicator function of an affine subspace

Thm: Existence of Highly-Biased Linear Approximations (2)

Let $(U + a) \subseteq \mathbb{F}_2^n$ be an invariant affine subspace for a permutation F. Then, for any nonzero $\gamma' \in U^\perp$, there exists a $\gamma \in U^\perp \setminus \{0\}$ such that

$$|\operatorname{cor}_F(\ell_\gamma, \ell_{\gamma'})| \geq 2^{-n+\dim U}$$

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In 2011, Leander et al. already proved the existence of a linear approximation with

$$|\operatorname{cor}_F(\ell_\gamma, \ell_{\gamma'})| \geq 2^{-n+\dim U} - 2^{2(-n+\dim U)} .$$

Open Questions

- Can we say anything more about the highly-biased linear approximations besides their mere existence?
- In particular, can we understand more about the distribution of the correlations $\mathrm{cor}_F(\ell_\gamma, \ell_{\gamma'})$ over all $\gamma, \gamma' \in \Gamma_g$?

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- we can now use linear cryptanalysis over $F^{\mathcal{G},\mathcal{G}^{-1}}$

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- we base the analysis on a single linear trail $(\alpha_0, \alpha_1, \ldots, \alpha_t)$ with correlation $\prod_{i=1}^{t} \mathrm{cor}_{R_{k_i}^{\mathcal{G},\mathcal{G}^{-1}}}(\ell_{\alpha_{i-1}}, \ell_{\alpha_i})$

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- choose a permutation $G \colon \mathbb{F}_2^4 \to \mathbb{F}_2^4$ with $g(x) = \langle 8, G(x) \rangle$ and define $\mathcal{G} \coloneqq (G, G, \ldots, G)$

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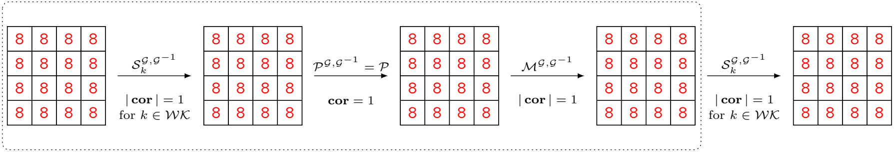

In this view, all S-boxes are active

A Four-Round Linear Trail for transformed Midori64

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- we omit the key-schedule of Midori64 and assume independent round keys!

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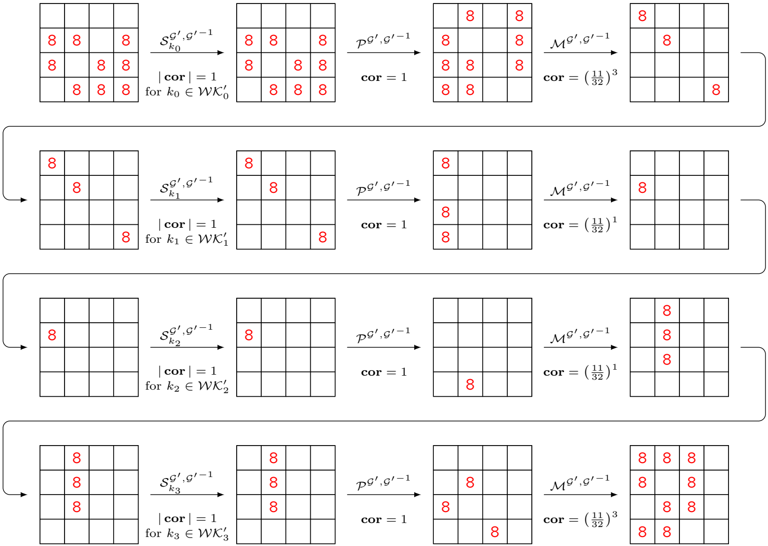

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- if we use independent round keys in each round, 2^{208} out of all possible 2^{256} keys are weak
- as the absolute correlation of the linear trail, we obtain $|\operatorname{cor}| = 2^{-12.325}$
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but

- by the wide-trail strategy, we expect $|\,\mathrm{cor}\,| \geq 2^{-16}$ as the correlation of a four-round linear trail (16 active S-boxes)

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$$|\operatorname{cor}_{S_k^{G', G'^{-1}}}(\ell_8, \ell_8)| = \begin{cases} 1 & \text{if } k \in \{(0,0,0,*)\} \\ \frac{1}{2} & \text{else} \end{cases}$$

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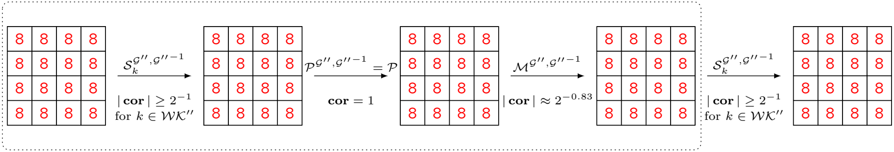

Correlation of the full-round trail is $\geq (2^{-1.83})^{16} = 2^{-29.28}$.

but..

A Strong Linear-Hull Effect

The trail correlation does not approximate the correlation of the approximation!

Ex: Single column

Let $\tilde{\mathcal{G}} = (G', G, G, G)$.

$$\begin{bmatrix}8\\8\\8\\8\end{bmatrix} \xrightarrow[|\,\mathbf{cor}\,| \geq 2^{-1}]{\mathcal{S}_{k_0}^{\tilde{\mathcal{G}},\tilde{\mathcal{G}}^{-1}}} \begin{bmatrix}8\\8\\8\\8\end{bmatrix} \xrightarrow[|\,\mathbf{cor}\,| \approx 2^{-0.83}]{\mathbf{M}^{\tilde{\mathcal{G}},\tilde{\mathcal{G}}^{-1}}} \begin{bmatrix}8\\8\\8\\8\end{bmatrix} \xrightarrow[|\,\mathbf{cor}\,| \geq 2^{-1}]{\mathcal{S}_{k_1}^{\tilde{\mathcal{G}},\tilde{\mathcal{G}}^{-1}}} \begin{bmatrix}8\\8\\8\\8\end{bmatrix} \xrightarrow[|\,\mathbf{cor}\,| \approx 2^{-0.83}]{\mathbf{M}^{\tilde{\mathcal{G}},\tilde{\mathcal{G}}^{-1}}} \begin{bmatrix}8\\8\\8\\8\end{bmatrix}$$

If $k_0 \in \mathbb{F}_2^4 \times \{(0,0,*,*)\}^3$ and $k_1 \in \left(\mathbb{F}_2^4 \setminus \{(0,0,*,*)\}\right) \times \{(0,0,*,*)\}^3$,

$$\mathrm{cor}_{\mathcal{R}_{k_1} \circ \mathcal{R}_{k_0}} \left(\ell_{(8,8,8,8)}, \ell_{(8,8,8,8)}\right) = 0$$

Open questions?

- In which cases can we approximate the approximation with a single trail?
- From another view: Can we use nonlinear approximations to quantify linear-hull effects in general?

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Thanks for your attention! Any questions?