Differentially 4-Uniform Permutations with the Best Known Nonlinearity from Butterflies

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- Low differential uniformity (to resist differential attacks);
- High nonlinearity (to resist linear attacks);
- Not too low algebraic degree (to resist higher order differential attacks or algebraic attacks).

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- its differential uniformity is 4;
- its nonlinearity is optimal (i.e., 112);
- its algebraic degree is optimal as well (i.e., 7).

Vectorial Boolean Functions

Definition (Vectorial Boolean Functions)

Let n and m be two positive integers, The functions from $\mathbb{F}_2^n$ to $\mathbb{F}_2^m$ are called (n, m)-functions or vectorial Boolean functions. Specially, when $m = 1$, we call these $(n, 1)$-functions Boolean functions.

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- An (n,m)-function has the following coordinate form:

$$\begin{aligned}&F(x_1,x_2,\cdots,x_n)\\=&(f_1(x_1,x_2,\cdots,x_n),f_2(x_1,x_2,\cdots,x_n),\cdots,f_m(x_1,x_2,\cdots,x_n)),\end{aligned}$$

where each coordinate $f_i(x_1,x_2,\cdots,x_n), 1\le i\le m$ is a Boolean function.

Algebraic Normal Form

Algebraic Normal Form (ANF)

An (n,m)-function F can be uniquely represented as an element of $\mathbb{F}_2^m[x_1,x_2,\cdots,x_n]/\langle x_1^2+x_1,x_2^2+x_2,\cdots,x_n^2+x_n\rangle$:

$$F(x)=\sum_{I\in\mathcal{P}(N)} a_I\left(\prod_{i\in I} x_i\right)=\sum_{I\in\mathcal{P}(N)} a_I x^I,$$

where $\mathcal{P}(N)$ denotes the power set of $N=\{1,2,\cdots,n\}$, and $a_I\in\mathbb{F}_2^m$.

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$$F(x) = \sum_{I \in \mathcal{P}(N)} a_I \left(\prod_{i \in I} x_i\right) = \sum_{I \in \mathcal{P}(N)} a_I x^I,$$

where $\mathcal{P}(N)$ denotes the power set of $N = \{1, 2, \cdots, n\}$, and $a_I \in \mathbb{F}_2^m$.

The algebraic degree of the function is by definition the global degree of its ANF:

$$\deg(F) = \max\{|I| : a_I \neq (0, 0, \cdots, 0); I \in \mathcal{P}(N)\}$$

Univariate Polynomial Representation

A second representation of (n,m)-functions when $m=n$

Any (n,n)-function F admits a unique univariate polynomial representation over $\mathbb{F}_{2^n}[x]/\langle x^{2^n}+x\rangle$, of degree at most 2^n-1:

$$F(x)=\sum_{i=0}^{2^n-1} c_i x^i, \quad c_i \in \mathbb{F}_{2^n}.$$

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$$F(x) = \sum_{i=0}^{2^n-1} c_i x^i, \quad c_i \in \mathbb{F}_{2^n}.$$

- The algebraic degree of F is equal to the maximum *2-weight* $w_2(i)$ of i such that $c_i \neq 0$, where $w_2(l)$ is the number of nonzero coefficients $l_j \in \mathbb{F}_2$ in the binary expansion $l = \sum_{j=0}^{n-1} l_j 2^j$.

Differential Uniformity

Definition (Differential Uniformity)

For a function $F : \mathbb{F}_{2^n} \to \mathbb{F}_{2^n}$, the differential uniformity of $F(x)$ is denoted as

$$\Delta_F = \max\{\delta_F(a, b) : a \in \mathbb{F}_{2^n}^*, b \in \mathbb{F}_{2^n}\},$$

where $\delta_F(a, b) = |\{x \in \mathbb{F}_{2^n} : F(x + a) + F(x) = b\}|$.

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- The *differential spectrum* of $F(x)$ is the multiset

$$\{* \ \delta_F(a,b) : a \in \mathbb{F}_{2^n}^*, b \in \mathbb{F}_{2^n} \ *\}.$$

Differential Uniformity

Obviously, if x_0 is a solution of $F(x+a)+F(x)=b$, so is x_0+a. Thus the differential uniformity must be even. The smallest possible value is 2. These functions which achieve this bound are called *almost perfect nonlinear (APN)* functions.

Examples

- Gold function x^{2^i+1}, $1 \le i \le \frac{n-1}{2}$, $\gcd(i,n)=1$ (Gold 1968);
- Kasami function $x^{2^{2i}-2^i+1}$, $1 \le i \le \frac{n-1}{2}$, $\gcd(i,n)=1$ (Kasami 1971);
- Welch function x^{2^t+3}, $n=2t+1$ (Niho 1972);
- ...

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A natural tradeoff method is to use differentially 4-uniform permutations as S-boxes. It is interesting to construct more differentially 4-uniform permutations with high nonlinearity and algebraic degree.

Nonlinearity

Walsh transform

For any function $F : \mathbb{F}_{2^n} \to \mathbb{F}_{2^n}$, we define the *Walsh transform* of F as

$$\mathcal{W}_F(a, b) = \sum_{x \in \mathbb{F}_{2^n}} (-1)^{\mathrm{Tr}(bF(x)+ax)}, \quad a, b \in \mathbb{F}_{2^n},$$

where $\mathrm{Tr}(x) = x + x^2 + \cdots + x^{2^{n-1}}$ is the absolute trace function from $\mathbb{F}_{2^n}$ to $\mathbb{F}_2$.

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The multiset $\Lambda_F = \{* \ \mathcal{W}_F(a,b) : a \in \mathbb{F}_{2^n}, b \in \mathbb{F}_{2^n}^* \ *\}$ is called the *Walsh spectrum* of the function F.

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The *nonlinearity* of F is defined as

$$\mathcal{NL}(F) = 2^{n-1} - \frac{1}{2} \max_{a \in \mathbb{F}_{2^n}, b \in \mathbb{F}_{2^n}^*} |\mathcal{W}_F(a,b)|.$$

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- If n is odd the nonlinearity of F satisfies the inequality $\mathcal{NL}(F) \leq 2^{n-1} - 2^{\frac{n-1}{2}}$, and in case of equality F is called *almost bent* function.
- While n is even, the known maximum nonlinearity is $2^{n-1} - 2^{\frac{n}{2}}$. It is conjectured that $\mathcal{NL}(F)$ is upper bounded by $2^{n-1} - 2^{\frac{n}{2}}$. These functions which meet this bound are usually called *optimal (maximal) nonlinear* functions.

Butterfly Structures

Definition (Butterfly Structures)

Let k be a positive integer and $\alpha \in \mathbb{F}_{2^k}$, e be an integer such that the mapping $x \mapsto x^e$ is a permutation over $\mathbb{F}_{2^k}$ and $R_z[e,\alpha](x) = (x+\alpha z)^e + z^e$ be a keyed permutation. The Butterfly Structures are defined as follows:

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- the *Open Butterfly Structure* with branch size k, exponent e and coefficient α is the function denoted H_e^α defined by:

$$\mathrm{H}_e^\alpha(x,y) = \left(R_{R_y^{-1}[e,\alpha](x)}[e,\alpha](y), R_y^{-1}[e,\alpha](x)\right),$$

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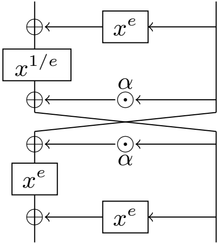
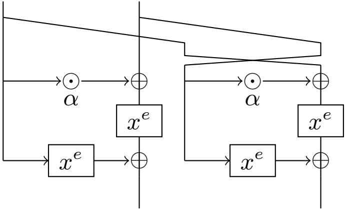
Butterfly Structures



(a) Open butterfly H_e^α (bijective).



(b) Closed butterfly V_e^α.

The Algebraic Forms of Butterfly Structures

- Open Butterfly Structure

$$\begin{aligned}&\mathsf{H}_e^{\alpha}(x,y)\\ &=\left(\left(y+\alpha(x+y^e)^{\frac{1}{e}}+\alpha^2 y\right)^e+\left((x+y^e)^{\frac{1}{e}}+\alpha y\right)^e,(x+y^e)^{\frac{1}{e}}+\alpha y\right)\end{aligned}$$

- Closed Butterfly Structure

$$\mathsf{V}_e^{\alpha}(x,y)=((\alpha x+y)^e+x^e,(x+\alpha y)^e+y^e)$$

Generalised Butterfly Structures

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Let R be a bivariate polynomials of $\mathbb{F}_{2^k}$ such that $R_y : x \mapsto R(x, y)$ is a permutation of $\mathbb{F}_{2^k}$ for all y in $\mathbb{F}_{2^k}$. The Generalised Butterfly Structures are defined as follows:

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- the *Closed Generalised Butterfly Structure* with branch size k is the function denoted V_R defined by:

$$\mathrm{V}_R(x, y) = (R(x, y), R(y, x)).$$

Generalised Butterfly Structures

(a) Open Generalised Butterfly H_R.

(b) Closed Generalised Butterfly V_R.

Figure: The Generalised Butterfly Structures.

Equivalence Relations

- Two functions $F, G : \mathbb{F}_{2^n} \to \mathbb{F}_{2^n}$ are called *extended affine equivalent (EA-equivalent)*, if $G(x) = A_1(F(A_2(x))) + A_3(x)$, where $A_1(x)$, $A_2(x)$ are affine permutations over $\mathbb{F}_{2^n}$ and $A_3(x)$ is an affine function over $\mathbb{F}_{2^n}$.

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- They are called *CCZ-equivalent (Carlet-Charpin-Zinoviev equivalent)* if there exists an affine permutation over $\mathbb{F}_{2^n} \times \mathbb{F}_{2^n}$ which maps $\mathcal{G}_F$ to $\mathcal{G}_G$, where $\mathcal{G}_F = \{(x, F(x)) : x \in \mathbb{F}_{2^n}\}$ is the graph of F, and $\mathcal{G}_G$ is the graph of G.

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- H_e^{α} (H_R) and V_e^{α} (V_R) are CCZ-equivalent.

The Motivation of This Research

Theorem (Perrin et al. CRYPTO'16)

Let V_e^α *and* H_e^α *respectively be the closed and open* $2k$*-bit butterflies with exponent* $e = 3 \times 2^t$ *for some* t*, coefficient* α *not in* $\{0, 1\}$ *and* k *odd. Then:*

1. V_e^α *is quadratic, and half of the coordinates of* H_e^α *have algebraic degree* k*, the other half have algebraic degree* $k + 1$*;*
2. *The differential uniformity of both* H_e^α *and* V_e^α *are at most equal to 4.*

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A Conjecture

The nonlinearity of butterfly structures of H_e^α and V_e^α operating on $2k$ bits are equal to $2^{2k-1} - 2^k$ for every odd k, $e = 3 \times 2^t$ and $\alpha \neq 0, 1$.

Cryptographic Properties of Generalised Butterflies

Theorem (Canteaut-Duval-Perrin, 2017, TIT)

The cryptographic properties of the generalised butterflies $\mathsf{V}_{\alpha,\beta}$ *and* $\mathsf{H}_{\alpha,\beta}$ *which are based on functions* $R : (x, y) \mapsto (x + \alpha y)^3 + \beta y^3$ *with* $\alpha, \beta \neq 0$ *are as follows:*

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3. *if* $\beta = (1 + \alpha)^3$ *then the differential uniformity is equal to* 2^{k+1}*, the nonlinearity is equal to* $2^{2k-1} - 2^{\frac{3k-1}{2}}$ *and the algebraic degree of* $\mathsf{H}_{\alpha,\beta}$ *is equal to* k*;*

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4. *otherwise, the differential uniformity is equal to 4, the nonlinearity is equal to* $2^{2k-1} - 2^k$ *and algebraic degree of* $\mathsf{H}_{\alpha,\beta}$ *is either* k *or* $k + 1$*. It is equal to* k *if and only if* $1 + \alpha\beta + \alpha^4 = (\beta + \alpha + \alpha^3)^2$*.*

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- The differential uniformity of both H_e^α and V_e^α are at most equal to 4, where $e = (2^i + 1) \times 2^t$, coefficient $\alpha \neq 0, 1$, k odd and $\gcd(i, k) = 1$;

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- We show that V_e^1 for $e = (2^i + 1) \times 2^t$ are permutations over $\mathbb{F}_{2^k} \times \mathbb{F}_{2^k}$.

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Theorem (Nontrivial Case)

For any $0 \le t \le k-1$, $0 \le i \le k-1$, $\gcd(k,i)=1$, $\alpha \in \mathbb{F}_{2^k}$, *and* $\alpha \neq 0,1$, *let* H_e^α *and* V_e^α *be the open and closed* $2k$*-bit butterfly structures with exponent* $e=(2^i+1)\times 2^t$ *and coefficient* α. *Then*

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For any $0 \le t \le k-1$, $0 \le i \le k-1$, $\gcd(k,i)=1$, $\alpha \in \mathbb{F}_{2^k}$, *and* $\alpha \neq 0, 1$, *let* H_e^α *and* V_e^α *be the open and closed* $2k$*-bit butterfly structures with exponent* $e = (2^i+1)\times 2^t$ *and coefficient* α. *Then*

1. V_e^α *has algebraic degree 2. The open butterfly* H_e^α *has algebraic degree* $k+1$;
2. *The differential uniformity of both* H_e^α *and* V_e^α *are at most equal to 4;*
3. *The nonlinearity of both* H_e^α *and* V_e^α *are equal to* $2^{2k-1}-2^k$, *namely, optimal, and their extended Walsh spectrum are* $\{0, 2^k, 2^{k+1}\}$.

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Theorem (Trivial Cases)

For any $0 \le t \le k-1$ *and* $0 \le i \le k-1$, $\gcd(i,k) = 1$, *let* H_e^1 *and* V_e^1 *be the open and closed* $2k$*-bit butterfly structures with exponent* $e = (2^i + 1) \times 2^t$ *and coefficient* $\alpha = 1$. *then*

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- Suppose k and i are two integers such that $\gcd(i,k)=1$. For any $c_1,c_2,c_3\in\mathbb{F}_{2^k}$ with not all zero, then the following equation

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- Suppose k is an odd integer and $\gcd(i,k)=1$. For any $c_1, c_2, c_3 \in \mathbb{F}_{2^k}$ with not all zero, then the following equation

$$c_1 x^{2^{4i}} + c_2 x^{2^{2i}} + c_3 x = 0$$

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The Proof of Differential Uniformity

Let $u, v, a, b \in \mathbb{F}_{2^k}$ and $(u, v) \neq (0, 0)$. Then we need to prove that

$$\mathsf{V}_e^{\alpha}(x, y) + \mathsf{V}_e^{\alpha}(x + u, y + v) = (a, b),$$

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$$\begin{cases} \left(\alpha^{2^i}(\alpha u + v) + u\right) x^{2^i} + \left(\alpha(\alpha u + v)^{2^i} + u^{2^i}\right) x \\ \qquad\qquad + (\alpha u + v) y^{2^i} + (\alpha u + v)^{2^i} y = 0, \\ (\alpha v + u) x^{2^i} + (\alpha v + u)^{2^i} x + \left(\alpha^{2^i}(\alpha v + u) + v\right) y^{2^i} \\ \qquad\qquad + \left(\alpha(\alpha v + u)^{2^i} + v^{2^i}\right) y = 0 \end{cases}$$

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The Proof of Nonlinearity

Let $a, b, c, d \in \mathbb{F}_{2^k}$, and $(c, d) \neq (0, 0)$. Then we have

$$\mathcal{W}_F^2((a,b),(c,d)) = \sum_{x,y \in \mathbb{F}_{2^k}} (-1)^{F(x,y)} \cdot \sum_{u,v \in \mathbb{F}_{2^k}} (-1)^{F(x+u,y+v)}$$

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&= \sum_{x,y,u,v \in \mathbb{F}_{2^k}} (-1)^{F(x,y)+F(x+u,y+v)} \\
&= 2^{2k} \cdot \sum_{u,v \in R(c,d)} (-1)^{f(u,v)},
\end{aligned}$$

where

$$\begin{aligned}
f(x,y) = &\mathrm{Tr}\Big((\alpha^{2^i+1} c + c + d) x^{2^i+1} + (\alpha^{2^i+1} d + c + d) y^{2^i+1} \\
&+ (\alpha^{2^i} c + \alpha d) x^{2^i} y + (\alpha c + \alpha^{2^i} d) x y^{2^i} + ax + by \Big),
\end{aligned}$$

The Proof of Nonlinearity

and $R(c,d)$ is the solution set of the following system of equations with variables u, v

$$\begin{cases} \left(\alpha^{2^i+1}c+c+d\right)^{2^i} u^{2^{2i}} + \left(\alpha^{2^i+1}c+c+d\right) u \\ \qquad + \left(\alpha c+\alpha^{2^i}d\right)^{2^i} v^{2^{2i}} + \left(\alpha^{2^i}c+\alpha d\right) v = 0, \\ \left(\alpha^{2^i}c+\alpha d\right)^{2^i} u^{2^{2i}} + \left(\alpha c+\alpha^{2^i}d\right) u \\ \qquad + \left(\alpha^{2^i+1}d+c+d\right)^{2^i} v^{2^{2i}} + \left(\alpha^{2^i+1}d+c+d\right) v = 0. \end{cases}$$

The core part: $\dim_{\mathbb{F}_2} R(c,d) = 0$ or 2.

The bijective of closed butterfly structure

For any $u, v \in \mathbb{F}_{2^k}$, where $(u, v) \neq (0, 0)$, it is sufficient to show that

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This is to say that the following system of equations

$$\begin{cases} vx^{2^i} + v^{2^i}x + (u+v)y^{2^i} + (u+v)^{2^i}y = (u+v)^{2^i+1} + u^{2^i+1}, \\ (u+v)x^{2^i} + (u+v)^{2^i}x + uy^{2^i} + u^{2^i}y = (u+v)^{2^i+1} + v^{2^i+1} \end{cases}$$

has no solution in $\mathbb{F}_{2^k} \times \mathbb{F}_{2^k}$.

The Nonlinearity

The proof procedure of the nonlinearity of trivial case is mainly based on the following lemma.

Lemma

Let i be an integer such that $0 \le i \le k-1$ and $\gcd(k,i) = 1$. Then for any $(c,d) \in \mathbb{F}_{2^k}^2$ with $(c,d) \neq (0,0)$, the following system of equations in variables u and v

$$\begin{cases} du^{2^i} + (du)^{2^{k-i}} + (c+d)v^{2^i} + ((c+d)v)^{2^{k-i}} = 0, \\ (c+d)u^{2^i} + ((c+d)u)^{2^{k-i}} + cv^{2^i} + (cv)^{2^{k-i}} = 0 \end{cases}$$

has exactly 4 solutions in $\mathbb{F}_{2^k} \times \mathbb{F}_{2^k}$.

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- We prove that their nonlinearities are optimal in a general case;
- We prove that the closed butterfly structure with trivial coefficient is also a permutation.

Open Problems

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- **The BIG APN problem**: Is there a tuple $k, R(x, y)$ where $k > 3$ is an integer, such that $\mathrm{H}_R(x, y)$ operating on $\mathbb{F}_{2^k} \times \mathbb{F}_{2^k}$ is APN?

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- Find more k, e, α where e is an integer and $\alpha \in \mathbb{F}_{2^k}$, such that H_e^α operating on $\mathbb{F}_{2^k} \times \mathbb{F}_{2^k}$ for even k is differential 4-uniform. (E.g., in the case $k = 6$ there does exist α such that H_5^α is differential 4-uniform)

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- Find more classes of differentially 4-uniform permutations with the optimal nonlinearity and high algebraic degree from **other functions over subfields or other structures**.

Thanks!